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Axially Symmetric, Spatially Homothetic Spacetimes

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We show that the existence of appropriate spatial homothetic Killing vectors is directly related to the separability of the metric functions for axially symmetric spacetimes. The density profile for such spacetimes is (spatially) arbitrary and admits *any* equation of state for the matter in the spacetime. When used for studying axisymmetric gravitational collapse, such solutions do not result in a locally naked singularity.

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The Newtonian notion of self-similarity [1] can be generalized in General Relativity in different ways [2]. It is important to distinguish between these generalizations.

In general, the notion of any self-similarity or scale-independence of the spacetime geometry requires the spacetime to admit a homothetic Killing vector. A proper homothetic Killing vector $\mathbf{X}$ satisfies

$$\mathcal{L}_{\mathbf{X}} g_{ab} \;=\; 2\,\Phi\, g_{ab} \tag{1}$$

where Φ is an arbitrary constant. This is also the broadest (Lie) sense of the scale invariance of the spacetime leading to the reduction of the Einstein field equations as partial differential equations to ordinary differential equations. In particular, a *spatially homothetic spacetime* admits an appropriate *spatial* homothetic Killing vector.

On the other hand, the notion of *self-similarity* of physical fields requires that the physical quantities transform according to their respective dimensions. The self-similarity property of matter fields then requires restrictions on the spacetime geometry. Such restrictions originate from the Einstein field equations through the energy-momentum tensor of the matter.

In general, the self-similarity of geometry need not imply self-similarity of physical fields. This leads to the question of whether the matter fields exhibit the same symmetries as the geometry - the so called "symmetry" inheritance problem. Moreover, the self-similarity of physical fields need not, in general, force the spacetime to admit a homothetic Killing vector. This is the general problem of obtaining the constraints on the metric from the self-similarity properties of matter fields - the so called "inverse" symmetry inheritance problem. In particular, we note that if the source of the spacetime is not perfect fluid, the spacetime symmetries need not be obeyed [3] by the matter fields. Needless to say, it is important to distinguish between all these cases.

We have recently shown [4] that the imposition of the admission of a *spatial* homothetic Killing vector

$$\bar{X}_a = (0, f(r,t), 0, 0) \tag{2}$$

on general spherically symmetric spacetimes forces those spacetimes to be necessarily separable in temporal and radial coordinates. The spacetime metric that admits (2) as a homothetic Killing vector is *uniquely* the one reported in [5], namely:

$$ds^2 = -\,y^2(r)\,dt^2 \;+\; \gamma^2\,B^2(t)\,(y')^2\,dr^2 \;+\; y^2(r)\,Y^2(t)\;\left[\,d\theta^2 \,+\, \sin^2\theta\, d\phi^2\,\right] \tag{3}$$

where an overhead prime denotes a derivative with respect to r, $f(r,t) = y/(\gamma y')$ and γ is a con-

stant. (We absorb a temporal function in g_{tt} by a redefinition of t.) We emphasize that the pure radial scale-independence requires the spherically symmetric spacetime to admit a *spatial* homothetic Killing vector (2).

The Einstein tensor for (3) has components

$$
\begin{aligned}
G_{tt} &= \frac{1}{Y^2} - \frac{1}{\gamma^2 B^2} + \frac{\dot{Y}^2}{Y} + 2\frac{\dot{B}\dot{Y}}{BY} && (4)\\
G_{rr} &= \gamma^2 B^2 \left(\frac{y'}{y}\right)^2 \left[-2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y} + \frac{3}{\gamma^2 B^2} + \frac{1}{Y^2}\right] && (5)\\
G_{\theta\theta} &= -Y\ddot{Y} - Y^2\frac{\ddot{B}}{B} - Y\frac{\dot{Y}\dot{B}}{B} + \frac{Y^2}{\gamma^2 B^2} && (6)\\
G_{\phi\phi} &= \sin^2\theta\, G_{\theta\theta} && (7)\\
G_{tr} &= 2\frac{\dot{B}y'}{By} && (8)
\end{aligned}
$$

Notice that G_{tr} component of the Einstein tensor is non-vanishing. Hence, matter in the spacetime could be *imperfect* or *anisotropic* indicating that the energy-momentum tensor could be

$$
\begin{aligned}
{}^{\mathrm{I}}T_{ab} &= (p + \rho)\, U_a U_b + p\, g_{ab} + q_a U_b + q_b U_a - 2\,\eta\,\sigma_{ab} && (9)\\
{}^{\mathrm{A}}T_{ab} &= \rho\, U_a U_b + p_{||}\, n_a n_b + p_\perp P_{ab} && (10)
\end{aligned}
$$

where U^a is the matter 4-velocity, q^a is the heat-flux 4-vector relative to U^a, η is the shear-viscosity coefficient, σ_{ab} is the shear tensor, n^a is a unit spacelike 4-vector orthogonal to U^a, P_{ab} is the projection tensor onto the two-plane orthogonal to U^a and n^a, $p_{||}$ denotes pressure parallel to and $p_\perp$ denotes pressure perpendicular to n^a. Also, p is the isotropic pressure and ρ is the energy density.

Now, the Einstein field equations with imperfect matter of vanishing shear viscosity yield for (3)

$$
\begin{aligned}
\rho &= \frac{1}{y^2}\left(\frac{\dot{Y}^2}{Y^2} + 2\frac{\dot{B}}{B}\frac{\dot{Y}}{Y} + \frac{1}{Y^2} - \frac{1}{\gamma^2 B^2}\right) && (11)\\
2\,\frac{\ddot{Y}}{Y} + \frac{\ddot{B}}{B} &= \frac{2}{\gamma^2 B^2} - \frac{y^2}{2}\,(\rho + 3\,p) && (12)\\
q &= -\frac{2\dot{B}}{y^2\gamma^2 y' B^3} && (13)
\end{aligned}
$$

where $q^a = (0, q, 0, 0)$ is the radial heat-flux vector. Therefore, the spacetime of (3) has the property that the temporal and radial metric functions are determined independently of each other. The Einstein field equations do not determine the radial function $y(r)$ [5]. On the other hand, the temporal functions $B(t)$ and $R(t)$ are determined by the properties of matter generating the spacetime such as its equation of state.

However, it is usual [6] to enforce on the spheri-

cally symmetric metric the form

$$\tilde{X}_a = (T, R, 0, 0) \tag{14}$$

for the homothetic Killing vector. Then, all the dimensionless quantities such as metric functions etc. are functions only of the dimensionless self-similarity variable T/R.

However, in [7] we showed that, for spherically symmetric spacetimes, the form (14) of the homothetic Killing vector is too restrictive and obscures important information about the properties of such spacetimes, for example, the existence of naked singularities. This is understandable since the Killing vector (14) corresponds to the *simultaneous* scale-invariance of the spacetime in T and R in the sense of Lie.

We emphasize that, for spherical symmetry, the appropriate form is (2) since it corresponds to only the radial scale-invariance of the spacetime in the sense of Lie. However, (2) is equivalent to (14) under the transformation

$$R = l(t)\exp\left(\int f^{-1}dr\right) \qquad T = k(t)\exp\left(\int f^{-1}dr\right) \tag{15}$$

Of course, there exists a relation between the temporal functions $l(t)$ and $k(t)$ since the transformed metric can always be made diagonal in R and T coordinates. It should be noted that spacetimes admitting (14) are included in (3) when the transformations (15) are non-singular and, hence, are preserving the spatial character of (2). Such non-singular transformations preserve the spatial scale-invariance property of gravity, in general.

Further, the spacetime of (3) admits [4] matter density that is an arbitrary function of the radial coordinate r since the field equations do not determine $y(r)$ and $\rho \propto 1/y^2$. This property of the spacetime, that of admitting an arbitrary spatial density, then also relates to the scale-invariance of the geometry in r. Hence, the homothetic Killing vector (2) indicates a fundamental property of gravity that it has no radial scale for matter inhomogeneities. Such spacetimes are then of fundamental importance to matters of gravity.

Clearly, any imposition of the homothetic Killing vector (14) on a spherically symmetric spacetime is *over-restrictive* and is not demanded by any *basic property of gravitation.* It can then be shown [8] that the spacetime of (3) does not lead to a naked singularity when the density is initially non-singular. It follows that the naked singularities can only arise when (15) are singular, that is when one of the basic properties of gravity, the pure spatial scale-invariance, is violated.

We note that a perfect fluid spacetime cannot admit a non-trivial homothetic Killing vector which is orthogonal to the fluid 4-velocity unless $p = \rho$ [9]. The spacetime of (3) admits (2) - a non-trivial, spatial homothetic Killing vector orthogonal to the fluid 4-velocity. When the temporal functions in (3) are constants, the equation of state for the matter, a perfect fluid now, is

$$p \;=\; \frac{1}{y^2}\left(\frac{4}{\gamma^2 B^2} \;-\; \frac{2}{Y^2}\right) \;+\; \rho \tag{16}$$

It is *uniquely* $p = \rho$ since B, Y, γ are arbitrary constants that can be chosen suitably.

Encouraged by the example in spherical symmetry, in our current paper, we then consider the implications of such a requirement of pure *spatial* homothety for axially symmetric spacetimes. In axial symmetry we have two spatial variables which can be expected to behave in a homothetic manner, viz. r and z. In other words, we expect the spacetime to admit arbitrary functions of r and z determining the matter characteristics of axially symmetric spacetimes.

Guided by these considerations, it would be natural to consider the existence of a homothetic Killing vector of the form

$$X_a = (0, f(r), g(z), 0) \tag{17}$$

for the axisymmetric metric

$$ds^2 = -\,\bar{A}^2(t,r,z)dt^2 + \bar{C}^2(t,r,z)dr^2 + \bar{D}^2(t,r,z)dz^2 + \bar{B}^2(t,r,z)d\phi^2 \tag{18}$$

However, the form (17) suggests a relationship between the r and z variables which is quite restrictive since it will correspond to *simultaneous* scale-invariance of these variables in the sense of Lie. This is contrary to our expectation that the matter characteristics in r and in z be independently specifiable. We expect the spacetime to be determined independently in these two spatial directions since gravity specifies no length-scale in either variables and not just for their combination. For example, the density distribution of a cylinder in r and z directions should, in general, be independently specifiable and that too in any desirable manner since gravity provides no length-scale for inhomogeneities in either variables.

Taking the above into account, we impose the existence of two independent *spatial* homothetic Killing vectors of the form

$$\mathbf{H}_r = (0, f(r), 0, 0) \tag{19}$$

and

$$\mathbf{H}_z = (0, 0, g(z), 0) \tag{20}$$

on (18). This reduces the metric (18) *uniquely* to

$$ds^2 = \exp\left[\left(\int \frac{\Phi_1 dr}{f(r)} + \int \frac{\Phi_2 dz}{g(z)}\right)\right]\left[A^2(t)dt^2 + \frac{C^2(t)}{f^2(r)}dr^2 + \frac{D^2(t)}{g^2(z)}dz^2 + B^2(t)d\phi^2\right] \tag{21}$$

where the Φ's are the constant conformal factors. Choosing $f(r) = \Phi_1 y(r)/y'$ and $g(z) = \Phi_2 Z(z)/\tilde{Z}$ where an overhead prime denotes differentiation with respect to r and an overhead tilde denotes differentiation with respect to z, we then obtain

$$ds^2 = -Z^2y^2dt^2 + \gamma_1^2 Z^2 C^2 (y')^2 dr^2 + \gamma_2^2 D^2 y^2 (\tilde{Z})^2 dz^2 + Z^2 y^2 B^2 d\phi^2 \tag{22}$$

where the γ's are constants related to Φ s. We also absorb a temporal function in g_{tt} by a redefinition of the time coordinate.

The Einstein tensor for (22) has the components

$$G_{tt} = -\frac{1}{\gamma_2^2 D^2} - \frac{1}{\gamma_1^2 C^2} + \frac{\dot{C}\dot{D}}{CD} + \frac{\dot{B}\dot{D}}{BD} + \frac{\dot{B}\dot{C}}{BC} \tag{23}$$

$$G_{rr} = \gamma_1^2 C^2 \left(\frac{y'}{y}\right)\left[-\frac{\ddot{D}}{D} - \frac{\ddot{B}}{B} - \frac{\dot{B}\dot{D}}{BD} + \frac{3}{\gamma_1^2 C^2} + \frac{1}{\gamma_2^2 D^2}\right] \tag{24}$$

$$G_{zz} = \gamma_2^2 D^2 \left(\frac{\tilde{Z}}{Z}\right)\left[-\frac{\ddot{C}}{C} - \frac{\ddot{B}}{B} - \frac{\dot{B}\dot{C}}{BC} + \frac{3}{\gamma_2^2 D^2} + \frac{1}{\gamma_1^2 C^2}\right] \tag{25}$$

$$G_{\phi\phi} = B^2\left[-\frac{\ddot{D}}{D} + \frac{1}{\gamma_2^2 D^2} - \frac{\ddot{C}}{C} - \frac{\dot{C}\dot{D}}{CD} + \frac{1}{\gamma_1^2 C^2}\right] \tag{26}$$

$$G_{tr} = 2\frac{\dot{C}y'}{Cy} \tag{27}$$

$$G_{tz} = 2\frac{\dot{D}\tilde{Z}}{DZ} \tag{28}$$

$$G_{rz} = 2\frac{\tilde{Z}y'}{Xy} \tag{29}$$

with an overhead dot denoting a time derivative. It is clear from the above that the spacetime necessarily possesses energy and momentum fluxes. The matter in the spacetime is *imperfect/anisotropic*.

This is interesting in its own right. Any mass-particle of an axisymmetric body has a Newtonian gravitational force directed along the line joining it to the origin. This force, which is unbalanced during the collapse, has generally non-vanishing components along r and z axes. Hence, a non-static axisymmetric spacetime of (22) will necessarily possess appropriate energy-momentum fluxes!

It is important to note that the coordinates (t, r, z, ϕ) are *not* co-moving. These coordinates are geared to the spatial homothetic Killing vectors. The matter 4-velocity, in general, will have all the four components, ie, $U^a = \left(U^t, U^r, U^z, U^\phi\right)$.

In the case that $U^\phi = 0$, the spacetime of (22) describes any non-rotating, axisymmetric matter configuration, in particular, a cigar configuration. In the case that $U^\phi \neq 0$, the spacetime of metric (22) describes *rotating* matter configurations. In other words, it represents the "internal" Kerr spacetimes that are also axisymmetric in nature. (It is also clear that non-static "internal" Kerr spacetimes cannot admit any perfect fluid matter since axisymmetry requires the existence of appropriate energy-momentum fluxes in such spacetimes as is evident from the earlier discussion.)

The spacetime (22) has a singularity when either $C(t) = 0$ or $D(t) = 0$ for some t or when $y(r) = 0$ for some r and/or $Z(z) = 0$ for some z. Moreover, from (22), the r and z null cone equations are

$$\frac{dt}{dr} = \pm\gamma_1 \frac{y'}{y} C(t) \tag{30}$$

$$\frac{dt}{dz} = \pm\gamma_2 \frac{\tilde{Z}}{Z} D(t) \tag{31}$$

and these are non-singular for nowhere-vanishing functions $y(r)$ and $Z(z)$. Hence, there does not exist an out-going null tangent at the spacetime singularity when $y(r) \neq 0$ and $Z(z) \neq 0$. Hence, the singularities of these axisymmetric spacetimes are *not* naked with these restrictions on the spatial functions.

We note that the nowhere-vanishing of $y(r)$ and $Z(z)$ means that the density is initially non-singular. Moreover, it is also clear that the spacetime of (22) will allow an arbitrary density profile in r and z since the field equations do not determine these spatial functions. Further, it is also seen that the spacetime of (22) admits *any* equation of state for the matter in the spacetime and that the properties of matter in the spacetime determine the temporal metric functions.

We also note that the energy-momentum tensor of the imperfect matter in the spacetime can contain contributions from the presence of electromagnetic fields in the matter. (That is why we listed only the Einstein tensor above.) The spacetime of (22) can then be used to describe the process of accretion of matter onto a rotating black hole. In this context, we note that the temporal behavior of the spacetime is all that is determinable from the properties of matter including those of the electromagnetic fields in the spacetime.

Moreover, a collapsing object could stabilize by the switching on of some forces opposing gravity. Stable such objects correspond to static spacetimes. Then, by considering temporal functions of the spatially homothetic spacetimes, namely, (3) and (22), appearing in the energy fluxes to be constants, we could obtain the spacetimes of stabilized objects with corresponding symmetries. When the equation of state of matter in a spatially homothetic, non-static spacetime changes to that of the corresponding static spacetime during the collapse, we obtain a stabilized object within these solutions. Note that we need to use appropriate form of the energy-momentum tensor, namely, (9) or (10), to stabilize the object.

Then, gravitational collapse may begin as dust but pressure must build up, nucleosynthesis may commence to produce heat and may result in a stabilized object like a star. The spatially homothetic spacetimes reported here accommodate these features because their temporal behavior is determined only by the properties of matter such as its equation of state.

Further, one could argue that time ought to behave in a self-similar manner as well and thus also impose a homothetic Killing vector of the form

$$\mathbf{H}_t = (h(t), 0, 0, 0) \tag{32}$$

However, this will impose a dynamic restriction on the spacetimes under consideration. For example, it forces the spherically symmetric spacetimes of (3) to be shear-free. (See [5, 10].)

Moreover, in the case of axial symmetry, we also see that all the dynamical and kinematical quantities are separable functions of t, r and z under the imposition of the homothetic Killing vectors (19) and (20). This is also the broadest (Lie) sense in which the field equations as partial differential equations reduce to ordinary differential equations and to their separable form.

Therefore, the imposition of appropriate *pure* spatial homothetic Killing vectors leads to spherically symmetric and axisymmetric separable metric spacetimes that admit any equation of state for *imperfect* and/or *anisotropic* matter. The matter

in these, generally non-static, spacetimes is imperfect and/or anisotropic since General Relativity as a theory of gravity seems to know, by virtue of its not providing any length-scale for matter properties, that the inhomogeneous collapsing matter will necessarily possess energy-momentum fluxes! The spacetimes obtained here admit an *arbitrary* spatial distribution of density. The spacetime singularity of such solutions is expected to result from their temporal evolution when the spatial distribution of density is initially non-singular. Such spacetime singularities are not locally naked. This is consistent with the Strong version of the Cosmic Censorship Hypothesis [11] which states that the singularities of gravitational collapse of matter with regular, non-singular initial data should not be visible to any observer, meaning that such singularities should not be locally naked.

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